A FINITENESS THEOREM FOR CANONICAL HEIGHTS ATTACHED TO RATIONAL MAPS OVER FUNCTION FIELDS

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Abstract. Let K be a function field, let $\varphi \in K(T)$ be a rational map of degree $d \geq 2$ defined over K, and suppose that φ is not isotrivial. In this paper, we show that a point $P \in \mathbb{P}^1(\bar{K})$ has φ-canonical height zero if and only if P is preperiodic for φ. This answers affirmatively a question of Szpiro and Tucker, and generalizes a recent result of Benedetto from polynomials to rational functions. We actually prove the following stronger result, which is a variant of the Northcott finiteness principle: there exists $\varepsilon > 0$ such that the set of points $P \in \mathbb{P}^1(K)$ with φ-canonical height at most ε is finite. Our proof is essentially analytic, making use of potential theory on Berkovich spaces to study the dynamical Green's functions $g_{\varphi,v}(x,y)$ attached to φ at each place v of K. For example, we show that every conjugate of φ has bad reduction at v if and only if $g_{\varphi,v}(x,x) > 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk},v}$, where $\mathbb{P}^1_{\mathrm{Berk},v}$ denotes the Berkovich projective line over the completion of $\bar{K}_v$. In an appendix, we use a similar method to give a new proof of the Mordell-Weil theorem for elliptic curves over K.

1. Introduction

1.1. **Terminology.** Throughout this paper, K will denote a *function field*, by which we will mean a field endowed with a set M_K of non-trivial non-archimedean absolute values which satisfies the *product formula*:

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(PF) For every nonzero $x \in K$, $|x|_v = 1$ for all but finitely many $v \in M_K$ and

$$\prod_{v \in M_K} |x|_v = 1 \ .$$

Examples of function fields include the field of rational functions on any normal (or just regular in codimension 1) projective variety over a field k (see [14],§2.3, and [9],§1.4.6).

The *field of constants* of a function field K is defined to be the subfield $k_0 \subset K$ consisting of all $x \in K$ such that $|x|_v \leq 1$ for all $v \in M_K$. By the product formula, if $x \in k_0$ is nonzero then in fact $|x|_v = 1$ for all $v \in M_K$.

1.2. **The canonical height attached to a rational map.** Let $\varphi \in K(T)$ be a rational map of degree $d \geq 2$ defined over K, so that φ acts on $\mathbb{P}^1(K)$ in the usual way. We define a *homogeneous lifting* of φ to be a choice of homogeneous polynomials $F_1, F_2 \in K[X, Y]$ of degree $d \geq 2$ having no common linear factor in $\bar{K}[X, Y]$ such that

$$\varphi([z_0 : z_1]) \ = \ [F_1(z_0, z_1) : F_2(z_0, z_1)]$$

for all $[z_0 : z_1] \in \mathbb{P}^1(K)$. (The polynomials F_1, F_2 are uniquely determined by φ up to multiplication by a common scalar $c \in K^*$.) The mapping

$$F = (F_1, F_2) : K^2 \to K^2$$

is a lifting of φ to K^2, and we denote by $F^{(n)} : K^2 \to K^2$ the iterated map $F \circ F \circ \ldots \circ F$ (n times).

The condition that F_1 and F_2 have no common linear factor over $\bar{K}$ can be rephrased by saying that the homogeneous resultant $\mathrm{Res}(F) = \mathrm{Res}(F_1, F_2)$ is nonzero. If $F_1(z) = \prod_i z \wedge \alpha_i$ and $F_2(z) = \prod_j z \wedge \beta_j$, then $\mathrm{Res}(F_1, F_2) = \prod_{i,j} \alpha_i \wedge \beta_j$, where $(x_0, x_1) \wedge (y_0, y_1) = x_0 y_1 - x_1 y_0$.

The canonical height function $\hat{H}_{F,K}$ is defined for $z \in K^2 \setminus \{0\}$ by

$$\hat{H}_{F,K}(P) = \lim_{n \to \infty} \frac{1}{d^n} \sum_{v \in M_K} \log \|F^{(n)}(P)\|_v \ ,$$

where $\|(x, y)\|_v = \max\{|x|_v, |y|_v\}$. It is straightforward to show using the product formula that $\hat{H}_{F,K}(z) \geq 0$, that $\hat{H}_{F,K}(z)$ depends only on the class of z in $\mathbb{P}^1(K)$, and that $\hat{H}_{cF,K} = \hat{H}_{F,K}$ for all $c \in K^*$. $\hat{H}_{F,K}$ therefore descends to a well-defined global canonical height function $\hat{h}_{\varphi,K} : \mathbb{P}^1(K) \to \mathbb{R}_{\geq 0}$ depending only on φ. For all $P \in \mathbb{P}^1(K)$, $\hat{h}_{\varphi,K}$ satisfies the functional equation

$$\hat{h}_{\varphi,K}(\varphi(P)) = d\hat{h}_{\varphi,K}(P) \ . \tag{1.1}$$

If K'/K is a finite extension, then for each place w of K' extending a given place v of K, one can define an absolute value on K' via $|x|_w = |N^{K'_w}_{K_v}(x)|_v$. The resulting set $M_{K'}$ of absolute values on K' again satisfies (PF), and for $P \in \mathbb{P}^1(K)$ we have

$$\hat{h}_{\varphi,K'}(P) = [K' : K]\hat{h}_{\varphi,K}(P) \ .$$

(See [18],§2.2, and [9], Proposition 1.4.2.) It follows that we can extend $\hat{h}_{\varphi,K}$ to $\bar{K}$ in a natural way by setting $\hat{h}_{\varphi,K}(P) = \frac{1}{[K':K]}\hat{h}_{\varphi,K'}(P)$ for $P \in \mathbb{P}^1(K')$. We will write $\hat{h}_\varphi(P)$ instead of $\hat{h}_{\varphi,K}(P)$ when no confusion is possible about which ground field we are talking about.

If $h_K : \mathbb{P}^1(K) \to \mathbb{R}_{\geq 0}$ denotes the standard Weil height on $\mathbb{P}^1(\bar{K})$, defined for $P = (z_0 : z_1) \in \mathbb{P}^1(K)$ by

$$h_K(P) = \sum_{v \in M_K} \log \max\{|z_0|_v, |z_1|_v\}$$

and extended to $\bar{K}$ as above, then there is a constant $C > 0$ such that

$$|\hat{h}_{\varphi,K}(P) - h_K(P)| \leq C \text{ for all } P \in \mathbb{P}^1(\bar{K}) \ . \tag{1.2}$$

This follows from (1.1) and the easily verified fact that

$$\hat{h}_{\varphi,K}(P) = \lim_{n\to\infty} \frac{1}{d^n} h_K(\varphi^{(n)}(P))$$

for all $P \in \mathbb{P}^1(\bar{K})$.

It is easy to see using (1.1) that $\hat{h}_{\varphi,K}(P) = 0$ if P is preperiodic for φ (i.e., if the orbit of P under iteration of φ is finite). And if the field of constants k_0 of K is finite, then just as in the number field case, it is easy to show that $\hat{h}_\varphi$ satisfies the following *Northcott finiteness property*: For any $M > 0$, the set

$$\{P \in \mathbb{P}^1(K) \ : \ \hat{h}_{\varphi,K}(P) \leq M\} \tag{1.3}$$

is finite. If property (1.3) holds for K, one easily deduces that if $\hat{h}_{\varphi,K}(P) = 0$, then P is preperiodic.

However, in the function field case, we have:

Lemma 1.4. *Let K be a function field. If the field of constants k_0 of K is infinite, then for M sufficiently large, the set*

$$\{P \in \mathbb{P}^1(K) \ : \ \hat{h}_{\varphi,K}(P) \leq M\} \tag{1.5}$$

is infinite.

Proof. If $P \in \mathbb{P}^1(k_0)$ then $h_K(P) = 0$, and therefore $\hat{h}_{\varphi,K}(P) \leq C$ for all $P \in \mathbb{P}^1(k_0)$ by (1.2). In particular, the set $\{P \in \mathbb{P}^1(K) \ : \ \hat{h}_{\varphi,K}(P) \leq C\}$ is infinite. ◻

In addition, if φ is defined over k_0, then $\hat{h}_{\varphi,K}(P) = 0$ for all $P \in \mathbb{P}^1(k_0)$, but not all points of $\mathbb{P}^1(k_0)$ are preperiodic. For example, if $k_0 = \bar{\mathbb{Q}}, K = \bar{\mathbb{Q}}(T)$, and $\varphi(T) = T^2$, then $\hat{h}_{\varphi,K}(P) = 0$ for all $P \in \mathbb{P}^1(\bar{\mathbb{Q}})$, but the only preperiodic points in $\mathbb{P}^1(K)$ (or $\mathbb{P}^1(\bar{K})$) are $0, \infty$, and the roots of unity in $\bar{\mathbb{Q}}$.

More generally, if φ is conjugate over a finite extension K' of K to a map defined over the field of constants of K', then (1.3) fails over K', since if $\varphi' = M^{-1} \circ \varphi \circ M$ with $M \in \mathrm{PGL}_2(K')$, then

$$\hat{h}_{\varphi',K'}(P) = \hat{h}_{\varphi,K'}(M(P))$$

for all $P \in \mathbb{P}^1(K')$ (see Lemma A.3 below).

We will show, however, that a weak version of (1.3) still holds over function fields, even when the field of constants is infinite. To state the result, define $\varphi \in K(T)$ to be *isotrivial over* K if there is a Möbius transformation $M \in \mathrm{PGL}_2(K)$ such that $\varphi' = M^{-1} \circ \varphi \circ M$ is defined over the field of constants of K, and *isotrivial* if there exists a finite extension K' of K such that φ is isotrivial over K'.

Theorem 1.6. *Let K be a function field, and let $\varphi \in K(T)$ be a rational map of degree $d \geq 2$. Assume that φ is not isotrivial. Then there exists $\varepsilon > 0$ (depending on K and φ) such that the set*

$$\{P \in \mathbb{P}^1(K) \ : \ \hat{h}_{\varphi,K}(P) \leq \varepsilon\}$$

is finite.

Remark 1.7.

(i) We do not know if Theorem 1.6 remains true if the hypothesis "φ is not isotrivial" is replaced by the *a priori* weaker hypothesis "φ is not isotrivial *over* K."

(ii) One cannot strengthen Theorem 1.6 to a Northcott-type theorem asserting, under the same hypotheses as Theorem 1.6, that for each $D \geq 1$, there exists $\varepsilon > 0$ (depending on D, K, and φ) such that the set

$$\{P \in \mathbb{P}^1(K') \ : \ [K' : K] \leq D \text{ and } \hat{h}_{\varphi,K}(P) \leq \varepsilon\}$$

is finite. For if this were true, then for all $n \geq 1$ the set

$$\{P \in \mathbb{P}^1(K) \ : \ \hat{h}_{\varphi,K}(P) \leq \varepsilon d^n\}$$

would be finite, since for each $P \in \mathbb{P}^1(K)$ there exists P' with $[K(P') : K] \leq d^n$ such that $\varphi^{(n)}(P') = P$ and $\hat{h}_{\varphi,K}(P) = d^n \hat{h}_{\varphi,K}(P')$. For n sufficiently large, this contradicts Lemma 1.4.

(iii) Combining Lemma 1.4 with Theorem 1.6, we see that if k_0 is infinite and φ is not isotrivial, then the quantity $\varepsilon_{\varphi,K} := \inf_{P \in \mathbb{P}^1(K)} \hat{h}_{\varphi,K}(P)$

is a positive real number. It would be interesting to investigate quantitatively the dependence of $\varepsilon_{\varphi,K}$ on φ and K.

As a consequence of (1.1) and Theorem 1.6, we obtain:

Corollary 1.8. *If K is a function field and $\varphi \in K(T)$ is a rational map of degree $d \geq 2$ which is not isotrivial, then a point $P \in \mathbb{P}^1(\bar{K})$ satisfies $\hat{h}_{\varphi,K}(P) = 0$ if and only if P is preperiodic for φ.*

Proof. As previously mentioned, preperiodic points always have height zero. Conversely, suppose $\hat{h}_{\varphi,K} = 0$ and choose a finite extension K' of K with $P \in \mathbb{P}^1(K')$. Using (1.1) and the fact that $\hat{h}_{\varphi,K'} = [K' : K]\hat{h}_{\varphi,K}$ for all $n \geq 0$, we obtain

$$\hat{h}_{\varphi,K'}(\varphi^{(n)}(P)) = [K' : K]\hat{h}_{\varphi,K}(\varphi^{(n)}(P)) = d^n[K' : K]\hat{h}_{\varphi,K}(P) = 0 \ .$$

By Theorem 1.6, it follows that the set $\{P, \varphi(P), \varphi^{(2)}(P), \ldots\}$ is finite, i.e., that P is preperiodic. □

The special case of Corollary 1.8 in which φ is a *polynomial map* was proved recently by Benedetto ([8], Theorem B). In fact, Benedetto proves a more precise result in the polynomial case ([8], Theorem A), as he just assumes that φ is not isotrivial *over K*. Corollary 1.8 answers affirmatively a question which we first learned of from Lucien Szpiro and Thomas Tucker.

Our proof of Theorem 1.6, which will be given in §2, uses the fact that isotriviality of φ can be detected in terms of good and bad reduction. In order to make this precise, we first recall the necessary definitions.

Let L be a valued field with valuation ring $\mathcal{O}_L$. If $\varphi \in L(T)$, we say that φ has *good reduction* over L if there exists a homogeneous lifting $F = (F_1, F_2)$ of φ having good reduction, i.e., $F_1, F_2 \in \mathcal{O}_L[x, y]$ and $\mathrm{Res}(F_1, F_2) \in \mathcal{O}_L^*$. If φ does not have good reduction over L, then we say that φ has *bad reduction* over L.

Also, we say that φ has *potentially good reduction* over L if there is a finite extension L' of L and a Möbius transformation $M \in \mathrm{PGL}_2(L')$ such that $\varphi' = M^{-1} \circ \varphi \circ M$ has good reduction over L'.

Conversely, we say that φ has *genuinely bad reduction* over L if it does not have potentially good reduction over L.

In §4, we will prove the following criterion for isotriviality:

Theorem 1.9. *Let K be a function field, and let $\varphi \in K(T)$ be a rational map of degree at least 2. Then φ is isotrivial if and only if φ has potentially good reduction over K_v for all $v \in M_K$, where K_v denotes the completion of K at v.*

1.3. **Dynamical Green's functions and reduction.** In order to apply Theorem 1.9 to canonical heights, we will use a decomposition

$$\hat{h}_{\varphi,K}(x) + \hat{h}_{\varphi,K}(y) = \sum_{v \in M_K} g_{\varphi,v}(x,y) \tag{1.10}$$

of the global canonical height into local contributions. Here $g_{\varphi,v}(x,y)$ is a two-variable *dynamical Green's function* attached to φ which was introduced in [5]; see §3 for a definition and further details. The function $g_{\varphi,v}(x,y)$ is defined first for x and y in the completion $\mathbb{C}_v$ of $\overline{K}_v$, and then extended in a natural way to the Berkovich projective line $\mathbb{P}^1_{\text{Berk}}$ over $\mathbb{C}_v$ (see §3.2).

In §3.4, we will use the theory of Berkovich spaces to prove a result characterizing genuinely bad reduction in terms of dynamical Green's functions. In order to state the result, let L be a complete and algebraically closed non-archimedean field, let $\varphi \in L(T)$ be a rational map of degree at least 2, and let $g_\varphi(x,y)$ be the corresponding dynamical Green's function on $\mathbb{P}^1(L)$. We view $\mathbb{P}^1(L)$ as being endowed with the analytic topology coming from the norm on L.

Theorem 1.11. *If φ has genuinely bad reduction, then there exists a constant $\beta > 0$ and a covering of $\mathbb{P}^1(L)$ by finitely many analytic open sets $V_1, \ldots, V_s$ such that for each $1 \leq i \leq s$, we have $g_\varphi(x,y) \geq \beta$ for all $x, y \in V_i$.*

As we will see, Theorem 1.11 can be easily deduced from the following result concerning the extension of $g_\varphi(x,y)$ to the Berkovich projective line:

Theorem 1.12. *φ has genuinely bad reduction over L if and only if $g_\varphi(x,x) > 0$ for all $x \in \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$.*

We will derive Theorem 1.12 as a special case of a more general result (Theorem 3.14) giving several different conditions which are all equivalent to φ having potentially good reduction. It is worth noting that Theorem 1.12, and therefore the theory of Berkovich spaces, is a crucial ingredient in our proof of Theorem 1.6, even though the statement of the latter result has nothing to do with Berkovich spaces.

Remark 1.13. The results in [4] are currently stated and proved in the special case where $L = \mathbb{C}_p$. However, all of the results from [4] relevant to the present paper remain valid over an arbitrary complete and algebraically closed non-archimedean field, see [6] and [21] for details.

Another (this time non-essential) ingredient in our proof of Theorem 1.6 is the following lower bound for average values of g_φ which was proved in [1], Theorem 1.1:

Theorem 1.14. *Let L be a valued field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Then there is an effective constant $\alpha > 0$, depending on φ and K, such that if $N \geq 2$ and $z_1, \ldots, z_N$ are distinct points of $\mathbb{P}^1(K)$, then*

$$\sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} g_\varphi(z_i, z_j) \geq -\alpha N \log N \ .$$

Remark 1.15.

(i) As noted above, Corollary 1.8 is a generalization of Benedetto's main theorem in [8] from polynomials to rational functions. As with the passage from the main result of [2] to that of [5], the rational function case of Corollary 1.8 seems to require more machinery than the polynomial case. For example, the analogue of Theorem 1.11 was proved in [8] by analyzing the radii of preimages of suitable disks under φ. However, we have not found a way to avoid using the machinery of Berkovich spaces to prove Theorem 1.11 for rational maps. Although the proofs of Theorems 1.9 and 1.14 do not rely on Berkovich's theory, they are also more technically involved than their polynomial counterparts in [8].

(ii) The interested reader should compare our results with those of Moriwaki [16], who obtains a Northcott-type theorem for varieties over function fields by replacing the usual Weil height (which he calls a "geometric height") with an "arithmetic height" coming from the choice of a certain kind of polarization.

(iii) It would be interesting to formulate and prove a result analogous to Theorem 1.6 in higher dimensions.

2. Proof of Theorem 1.6

We now give the proof of our main theorem on canonical heights over function fields (Theorem 1.6), assuming Theorems 1.9 and 1.11.

Proof. (Compare with Theorem 1.14 of [1].)

Recall from (1.10) that for all $z, w \in \mathbb{P}^1(K)$ with $z \neq w$, we have

$$\sum_{v \in M_K} g_{\varphi,v}(z, w) = \hat{h}_\varphi(z) + \hat{h}_\varphi(w) \ .$$

Since φ is not isotrivial, it follows from Theorem 1.9 that there exists a place $v_0 \in M_K$ at which φ has genuinely bad reduction. Let $\mathbb{C}_{v_0}$ be the smallest complete and algebraically closed field containing K_{v_0}. By Theorem 1.11, there is a finite covering $V_1, \ldots, V_s$ of $\mathbb{P}^1(\mathbb{C}_{v_0})$ and a constant $C_1 > 0$ such that $g_{\varphi,v_0}(z, w) \geq C_1$ whenever $z, w \in V_i$

$(i = 1, \ldots, s)$. If $z_1, \ldots, z_N \in \mathbb{P}^1(K)$, then by Theorem 1.14 and the fact that $g_{\varphi,v} \geq 0$ whenever φ has good reduction at v, there is a constant $C_2 > 0$ depending on φ and K such that

$$g(z_1, \ldots, z_N) := \sum_{v \in M_K} \sum_{i \neq j} g_{\varphi,v}(z_i, z_j) \geq -C_2 N \log N \ .$$

Moreover, if $z_1, \ldots, z_N \in V_i$ for some i, then

$$g(z_1 \ldots, z_N) \geq C_1 N^2 - C_2 N \log N \ . \tag{2.1}$$

Let $M = [\frac{N-1}{s}] + 1$. By the pigeonhole principle, in any subset of $\mathbb{P}^1(\mathbb{C}_{v_0})$ of cardinality N, there is an M-element subset contained in some V_i. Without loss of generality, order the z_j's so that this subset is $\{z_1, \ldots, z_M\}$. Applying (1.10) and (2.1), we obtain

$$C_1 M^2 - C_2 M \log M \leq g(z_1, \ldots, z_M) \leq 2M^2 \max_j \hat{h}_\varphi(z_j) \ .$$

If $\hat{h}_\varphi(z_j) \leq \frac{C_1}{4}$ for all $j = 1, \ldots, N$, then we obtain

$$\frac{C_1}{2} M \leq C_2 \log M \ ,$$

which implies that $M \leq C_3$ for some constant $C_3 > 0$ depending only on φ and K. Thus $N \leq Ms + 1 \leq C_4$ for some $C_4 > 0$ depending only on φ and K. Setting $\varepsilon = \frac{C_1}{4}$ now gives the desired result. $\square$

Remark 2.2. In the proof of Theorem 1.6, one could avoid appealing to Theorem 1.14 by applying Theorem 1.11 and a pigeonhole principle argument simultaneously at each place of genuinely bad reduction for φ. This would provide a better value of ε but worse upper bound on N than is given by the argument above.

3. Dynamical Green's functions, the Berkovich projective line, and reduction of rational maps

In this section, we study the dynamical Green's function $g_\varphi(z, w)$ attached to a rational map φ of degree $d \geq 2$ defined over a valued field L. A proof of Theorem 1.11 is given in §3.4.

3.1. **Definition and basic properties of dynamical Green's functions.** We begin by recalling some terminology from [1] and [5]. As in §1, write φ in the form

$$\varphi([z_0 : z_1]) \ = \ [F_1(z_0, z_1) : F_2(z_0, z_1)]$$

for some homogeneous polynomials $F_1, F_2 \in L[x, y]$ of degree $d \geq 2$ with $\mathrm{Res}(F_1, F_2) \neq 0$, and let $F = (F_1, F_2) : L^2 \to L^2$.

Write $\|(z_0, z_1)\| = \max\{|z_0|, |z_1|\}$, and for $z \in L^2\backslash\{0\}$, define the *homogeneous local dynamical height* $\hat{H}_F : L^2\backslash\{0\} \to \mathbb{R}$ by

$$\hat{H}_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^{(n)}(z)\| \ .$$

By convention, we set $\hat{H}_F(0,0) = -\infty$. According to [5], Lemma 3.5, the limit $\lim_{n\to\infty} \frac{1}{d^n} \log \|F^{(n)}(z)\|$ exists for all $z \in L^2\backslash\{0\}$, and $\frac{1}{d^n} \log \|F^{(n)}(z)\|$ converges uniformly on $L^2\backslash\{0\}$ to $\hat{H}_F(z)$.

The function $\hat{H}_F$ *scales logarithmically*, in the sense that for every $\alpha \in L^*$ and $z \in L^2\backslash\{0\}$,

$$\hat{H}_F(\alpha z) = \hat{H}_F(z) + \log|\alpha| \ . \tag{3.1}$$

If the value group $\{-\log|z| \ : \ z \in L^*\}$ is dense in $\mathbb{R}$ (which will be the case, for example, if L is algebraically closed and $|\cdot|$ is non-trivial), it follows from (3.1) that given $\varepsilon > 0$, for every $z \in L^2\backslash\{0\}$ there exists $\alpha \in L^*$ such that

$$-\varepsilon < \hat{H}_F(\alpha z) < 0 \ . \tag{3.2}$$

When $z, w \in L^2$ are linearly independent over L, define

$$G_F(z, w) \ = \ -\log|z \wedge w| + \hat{H}_F(z) + \hat{H}_F(w) + \log R \ ,$$

where $R = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}}$.

According to [5], Lemma 3.21, for all $\alpha, \beta, \gamma \in L^*$, we have

$$G_{\gamma F}(\alpha z, \beta w) = G_F(z, w) \ .$$

In particular, G_F descends to a well-defined function $g_\varphi(z, w)$ on $\mathbb{P}^1(L)$: for $z, w \in \mathbb{P}^1(L)$ and any lifts $\tilde{z}, \tilde{w} \in L^2\backslash\{0\}$,

$$g_\varphi(z, w) \ = \ -\log|\tilde{z} \wedge \tilde{w}| + \hat{H}_F(\tilde{z}) + \hat{H}_F(\tilde{w}) + \log R \ . \tag{3.3}$$

If $z \neq w$ then the right-hand side of (3.3) is finite; if $z = w$ then we set $g_\varphi(z, z) = +\infty$.

If K is a field satisfying (PF) and $\varphi \in K(T)$ is a rational map of degree $d \geq 2$, then for each $v \in M_K$ we have (setting $L = \mathbb{C}_v$) an associated dynamical Green's function $g_{\varphi,v}(z, w)$. By the product formula (applied twice), for all $z, w \in \mathbb{P}^1(K)$ with $z \neq w$ we have the fundamental identity

$$\sum_{v \in M_K} g_{\varphi,v}(z, w) = \hat{h}_\varphi(z) + \hat{h}_\varphi(w) \ .$$

3.2. **Extending the dynamical Green's function to Berkovich space.** There is a natural extension of $g_\varphi(z,w)$ to the Berkovich projective line; we briefly recall the relevant background material from [4],[6].

Let L be a complete non-archimedean field. With its usual topology, the projective line $\mathbb{P}^1(L)$ is totally disconnected, and if the residue field of L is infinite then $\mathbb{P}^1(L)$ is not locally compact. The Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk},L}$ over L is a connected compact Hausdorff space which contains $\mathbb{P}^1(L)$ as a dense subspace. The construction of $\mathbb{P}^1_{\mathrm{Berk}}$ is functorial, and in particular a rational map $\varphi : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$ extends in a natural way to a continuous map from $\mathbb{P}^1_{\mathrm{Berk}}$ to itself.

The space $\mathbb{P}^1_{\mathrm{Berk}}$ has the structure of a metric $\mathbb{R}$-tree (in the sense of [12]), and admits a rich *potential theory*, including a theory of harmonic and subharmonic functions. These notions are defined in terms of a measure-valued Laplacian operator Δ on $\mathbb{P}^1_{\mathrm{Berk}}$ whose domain is the space $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ of functions of *bounded differential variation* (see [4], §5.3). A similar theory has been developed independently by at least three different sets of authors – see [12], [4], and [21]. For simplicity, we will use [4] and [6] as our basic reference for potential theory on $\mathbb{P}^1_{\mathrm{Berk}}$, although the results we need are also contained in [21].

We recall from [5] that an Arakelov Green's function on the Berkovich projective line is a function $g(z,w) : \mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ such that

(B1) (Semicontinuity) The function $g(z,w)$ is lower-semicontinuous, and is finite and continuous off the diagonal.

(B2) (Differential equation) For each $w \in \mathbb{P}^1_{\mathrm{Berk}}$, $g(z,w)$ belongs to the space $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$. Furthermore, there is a probability measure μ on $\mathbb{P}^1_{\mathrm{Berk}}$ such that for each w, $g(z,w)$ satisfies the identity

$$\Delta_z g(z,w) \ = \ \delta_w(z) - \mu(z) \ .$$

Conditions (B1) and (B2) imply that $g(z,w)$ is symmetric and bounded below (see [4], Proposition 7.19). The semicontinuity along the diagonal is a technical condition which arises naturally from properties of the space $\mathbb{P}^1_{\mathrm{Berk}}$ (see [4], Proposition 3.1). Together, (B1) and (B2) determine $g(z,w)$ up to an additive constant by the maximum principle ([4], Proposition 5.14). If in addition

(B3) (Normalization) $\iint_{\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}} g(z,w)\, \mu(z)\mu(w) \ = \ 0$,

we will say $g(z,w)$ is a *normalized* Arakelov Green's function.

Since it is known that

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}} g(z,w)\mu(z) \equiv C \tag{3.4}$$

for some constant C independent of w (see [4], Proposition 7.24), (B3) is in fact equivalent to the *a priori* stronger condition

$$(\mathrm{B3})' \qquad \int g(z,w)\,\mu(z) \;\equiv\; 0 \;\;.$$

If $\varphi \in L(T)$ is a rational function of degree $d \geq 2$, then as explained in [5], §7.5, the dynamical Green's function g_φ extends in a natural way to a normalized Arakelov Green's function

$$g_\varphi(z,w) : \mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\} \;. \tag{3.5}$$

Explicitly, we have

$$g_\varphi(z,w) = \liminf_{\substack{(z_0,w_0)\in\mathbb{P}^1(L)\times\mathbb{P}^1(L)\\ (z_0,w_0)\to(z,w)}} g_\varphi(z_0,w_0) \;.$$

It is precisely because of the constant $\log R$ which appears in the definition of g_φ that it satisfies (B3); see Appendix A for details.

We will refer to the probability measure μ_φ on $\mathbb{P}^1_{\mathrm{Berk}}$ defined by the differential equation $\Delta_z g_\varphi(z,w) = \delta_w - \mu_\varphi$ as the *canonical measure* associated to φ. The fact that μ_φ is a *probability* measure (i.e., that μ_φ is nonnegative and has total mass 1) is proved in [4], Theorem 7.14. It is shown in [4], Proposition 7.15 that the μ_φ has no point masses on $\mathbb{P}^1(L)$. However, it can have point masses on $\mathbb{P}^1_{\mathrm{Berk}}\backslash\mathbb{P}^1(L)$ (see Theorem 3.14).

Lemma 3.6. *For each fixed $w \in \mathbb{P}^1_{\mathrm{Berk}}$, the function $f_w(z) = g_\varphi(z,w)$ is continuous on all of $\mathbb{P}^1_{\mathrm{Berk}}$ as an extended real-valued function. If $w \in \mathbb{P}^1_{\mathrm{Berk}}\backslash\mathbb{P}^1(L)$, then $f_w(z)$ is real-valued on all of $\mathbb{P}^1_{\mathrm{Berk}}$, and if $w \in \mathbb{P}^1(L)$, then $f_w(z)$ is real-valued except at $z = w$, where $f_w(w) = +\infty$.*

Proof. See Proposition 7.19 of [4]. □

There is a notion of subharmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$ (see [4], §6, or [21], §3.1.2, for a definition). We will use the fact that subharmonic functions satisfy the following *maximum principle*:

Theorem 3.7 (Maximum Principle for Subharmonic Functions)**.** *If U is an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$ and $f : U \to \mathbb{R} \cup \{+\infty\}$ is a subharmonic function which attains its maximum value on U, then f is locally constant.*

Proof. See [4], Proposition 6.15, and also [21], Proposition 3.1.11. □

As a consequence, we can identify where the minimum and maximum values of g_φ occur:

Theorem 3.8. *For each fixed $w \in \mathbb{P}^1_{\mathrm{Berk}}$, the minimum value of $f_w(z) = g_\varphi(z,w)$ is achieved on* $\mathrm{Supp}(\mu_\varphi)$*, and the maximum value is achieved at w.*

Proof. By Lemma 3.6, $f_w(z)$ attains its maximum and minimum values on $\mathbb{P}^1_{\text{Berk}}$. (The maximum value is $+\infty$ if $w \in \mathbb{P}^1(L)$.) By Proposition 6.1, Definition 6.2, and Proposition 7.19 of [4], $f_w(z)$ is subharmonic on $\mathbb{P}^1_{\text{Berk}}\backslash\{w\}$ and $-f_w(z)$ is subharmonic on the complement of $\text{Supp}(\mu_\varphi)$. (This can also be deduced from [21], Proposition 3.4.4.) The result therefore follows from the maximum principle for subharmonic functions (Theorem 3.7). □

Remark 3.9. Theorem 3.8 holds more generally for Arakelov-Green's functions associated to "log-continuous" probability measures, see [4], §7.

3.3. **Homogeneous filled Julia sets and transfinite diameters.** Let L be a complete valued field, and as in the previous section, let $F = (F_1, F_2)$ be a non-degenerate homogeneous polynomial mapping of L^2, i.e., $F_1, F_2 \in L[x, y]$ are homogeneous polynomials of degree $d \geq 2$ such that $\text{Res}(F) \neq 0$.

The *homogeneous filled Julia set* K_F of F in L^2 is the set of all $z \in L^2$ for which $\|F^{(n)}(z)\|$ stays bounded as n tends to infinity. Clearly $F(K_F) = K_F$ and $F^{-1}(K_F) = K_F$, i.e., K_F is completely invariant under F. According to [5], Lemma 3.8,

$$K_F = \{z \in L^2 \ : \ \hat{H}_F(z) \leq 0\} \ .$$

If F has good reduction, then by [5], Lemma 3.9, $K_F = B(0, 1)$ is the unit polydisc in L^2 and $\hat{H}_F(z) = \log \|z\|$ for all $z \in L^2$. We will establish a converse to this result shortly (Corollary 3.13).

Also, when F has good reduction, the dynamical Green's function is given quite simply by

$$G_F(z, w) = -\log |z \wedge w| + \log \|z\| + \log \|w\| \ . \tag{3.10}$$

Note that the right-hand side of (3.10) is always non-negative. This means that $g_\varphi(x, y) \geq 0$ for all $x, y \in \mathbb{P}^1(L)$ when φ has good reduction. By Lemma A.3, the same is true when φ has potentially good reduction. Conversely, we will see in Theorem 3.14 that if $g_\varphi(x, y) \geq 0$ for all $x, y \in \mathbb{P}^1(L)$, then φ has potentially good reduction.

Recall that for $z = (z_1, z_2), w = (w_1, w_2) \in L^2$, we have $z \wedge w = z_1w_2 - z_2w_1$. By analogy with the classical transfinite diameter, if $E \subset L^2$ is a bounded set, we define

$$d_n^0(E) \ = \ \sup_{z_1,\ldots,z_n \in E} \left(\prod_{i \neq j} |z_i \wedge z_j| \right)^{\frac{1}{n(n-1)}}$$

By [5], Lemma 3.10, the sequence of nonnegative real numbers $d_n^0(E)$ is non-increasing. In particular, the quantity $d_\infty^0(E) = \lim_{n\to\infty} d_n^0(E)$ is well-defined. We call $d_\infty^0(E)$ the *homogeneous transfinite diameter* of E.

We also define $d^0(E)$ to be

$$d^0(E) = \sup_{z,w\in E} |z \wedge w| \ .$$

We call $d^0(E)$ the *wedge diameter* of E.

As an example, it is easy to see that if L is a complete and algebraically closed non-archimedean field and $r \in |L^*|$, then the wedge diameter and homogeneous transfinite diameter of the polydisc

$$B(0,r) = \{z \in L^2 \ : \ \|z\| \le r\}$$

are both equal to r^2.

If $M \in \mathrm{GL}_2(L)$, then it is elementary to show that $|M(z)\wedge M(w)| = |\det(M)||z\wedge w|$, and thus $d_\infty^0(M(E)) = |\det(M)|d_\infty^0(E)$ and $d^0(M(E)) = |\det(M)|d^0(E)$.

As explained in [1] (see also [5]), for any valued field L we have

$$d_\infty^0(K_F) \ \le \ |\operatorname{Res}(F)|^{-1/d(d-1)} \ . \tag{3.11}$$

If L is complete and algebraically closed, then equality holds in (3.11) by Corollary A.16.

Lemma 3.12. *If L is a complete and algebraically closed non-archimedean field and $F(B(0,1)) \subseteq B(0,1)$, then all coefficients of F_1 and F_2 lie in $\mathcal{O}_L$.*

Proof. Write $F(x,y) = (a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d, b_0x^d + b_1x^{d-1}y + \cdots + b_dy^d)$. Since $F(B(0,1)) \subseteq B(0,1)$, in particular the one-variable polynomials $a(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$ and $b(x) = b_0x^d + b_1x^{d-1} + \cdots + b_d$ map $\mathcal{O}_L$ to itself. But then we are done, since

$$\sup_{z\in\mathcal{O}_L} |a(z)| = \max_{0\le i\le d} |a_i| \ ,$$

and similarly for $\sup_{z\in\mathcal{O}_L} |b(z)|$. $\square$

Corollary 3.13. *If L is a complete and algebraically closed non-archimedean field and $K_F = B(0,1)$, then F has good reduction.*

Proof. Since $K_F = B(0,1)$ implies that $F(B(0,1)) \subseteq B(0,1)$, it follows from Lemma 3.12 that all coefficients of F lie in $\mathcal{O}_L$. Also, $1 = d_\infty^0(K_F) = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}}$, so $|\operatorname{Res}(F)| = 1$. It follows that F has good reduction. $\square$

3.4. **Dynamical Green's functions and reduction of rational maps.** The main result of this section is the following theorem:

Theorem 3.14. *Let L be a complete and algebraically closed non-archimedean field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Then $g_\varphi(x,x) \geq 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, and the following are equivalent:*

(1) *φ has potentially good reduction.*
(2a) *$g_\varphi(x,y) \geq 0$ for all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$.*
(2b) *$g_\varphi(x,y) \geq 0$ for all $x, y \in \mathbb{P}^1(L)$.*
(3a) *$g_\varphi(\zeta,\zeta) = 0$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$.*
(3b) *$g_\varphi(\zeta,\zeta) = 0$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$.*
(4a) *$\mu_\varphi = \delta_\zeta$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$.*
(4b) *$\mu_\varphi = \delta_\zeta$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$.*

Note that conditions (2a) and (2b) are equivalent by the lower semicontinuity of $g_\varphi(x,y)$, (3a) and (3b) are equivalent because $g_\varphi(x,x) = +\infty$ for all $x \in \mathbb{P}^1(L)$, and (4a) and (4b) are equivalent because, as mentioned in §2.2, μ_φ has no point masses in $\mathbb{P}^1(L)$.

As an immediate corollary of Theorem 3.14, we have:

Corollary 3.15. *Under the same hypotheses as Theorem 3.14, the following are equivalent:*

(1) *φ has genuinely bad reduction.*
(2) *$g_\varphi(x,y) < 0$ for some $x, y \in \mathbb{P}^1(L)$.*
(3) *$g_\varphi(\zeta,\zeta) > 0$ for all $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$.*
(4) *μ_φ is not a point mass.*

Assuming Corollary 3.15, we show how to deduce Theorem 1.11, which is needed for the proof of Theorem 1.6.

Proof of Theorem 1.11. (Compare with [1], Lemma 3.1.)

Suppose φ has genuinely bad reduction over L. We claim that there exists a constant $\beta > 0$ such that $g_\varphi(x,x) > \beta$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. Indeed, according to Corollary 3.15, the fact that φ has genuinely bad reduction means that $g_\varphi(x,x) > 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. The claim then follows from the fact that $\mathbb{P}^1_{\mathrm{Berk}}$ is compact and g_φ is lower semicontinuous (see [10], § IV.6.2, Theorem 3). The lower semicontinuity of $g_\varphi(z,w)$, together with the definition of the product topology on $\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}$, also shows that for each $x \in \mathbb{P}^1_{\mathrm{Berk}}$, there is an open neighborhood U_x of x in $\mathbb{P}^1_{\mathrm{Berk}}$ such that $g_\varphi(z,w) > \beta$ for $z, w \in U_x$. By compactness of $\mathbb{P}^1_{\mathrm{Berk}}$, the covering $\{U_x \mid x \in \mathbb{P}^1_{\mathrm{Berk}}\}$ has a finite subcover $U_1, \ldots, U_s$. Take $V_i = U_i \cap \mathbb{P}^1(L)$.

□

Before giving the proof of Theorem 3.14, we establish some useful lemmas.

Lemma 3.16. *If L is an algebraically closed non-archimedean field and φ has potentially good reduction over L, then for any homogeneous lifting F of φ, there exists $M \in \mathrm{GL}_2(L)$ such that $M^{-1} \circ F \circ M$ has good reduction.*

Proof. Choose any $M' \in \mathrm{GL}_2(L)$ such that $\varphi' = M'^{-1} \circ \varphi \circ M'$ has good reduction. This means that there exists a homogeneous lifting F' of φ' with good reduction. Since any two lifts of φ' differ by a nonzero constant, we have $F' = c(M'^{-1} \circ F \circ M')$ for some $c \in L^*$. It is enough to prove that there exists $\alpha \in L^*$ such that $F' = M^{-1} \circ F \circ M$, where $M = \alpha M'$. For this, we take α such that $\alpha^{d-1} = c$, and then compute that

$$\begin{aligned}(M^{-1} \circ F \circ M)(z) &= \alpha^{-1}(M'^{-1} \circ F \circ M')(\alpha z) \\ &= \alpha^{d-1}(M'^{-1} \circ F \circ M')(z) \\ &= F'(z)\end{aligned}$$

as desired. □

Lemma 3.17. *Let L be an algebraically closed field, let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$, and let $F = (F_1, F_2)$ be a homogeneous lifting of φ. Then the natural map from the set of preperiodic points of F in $L^2 \backslash \{0\}$ to the set of preperiodic points of φ in $\mathbb{P}^1(L)$ is surjective.*

Proof. Let $P \in \mathbb{P}^1(L)$ be a preperiodic point of φ, so that $\varphi^{(j)}(P) = \varphi^{(k)}(P)$ for some positive integers $j \neq k$. Then for any homogeneous lifting $z \in L^2 \backslash \{0\}$ of P, there exists a nonzero constant $c \in L$ such that $F^{(j)}(z) = cF^{(k)}(z)$. Choose $\alpha \in L^*$ such that $\alpha^{d^k} = c\alpha^{d^j}$. Then

$$F^{(j)}(\alpha z) = \alpha^{d^j} F^{(j)}(z) = c\alpha^{d^j} F^{(k)}(z) = \alpha^{d^k} F^{(k)}(z) = F^{(k)}(\alpha z) \ ,$$

so that $\alpha z \in L^2$ is a lifting of P which is preperiodic for F. □

Lemma 3.18. *Let L be an algebraically closed field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Let $w, w' \in \mathbb{P}^1(L)$ be distinct preperiodic points of φ. Then there exists a Möbius transformation $M \in \mathrm{PGL}_2(L)$ with $M(\infty) = w$ and $M(0) = w'$, and a lifting $F' = (F_1', F_2')$ of $\varphi' = M^{-1} \circ \varphi \circ M$ such that $(1, 0)$ and $(0, 1)$ are preperiodic points for F'. (Here we identify the point $(1 : 0) \in \mathbb{P}^1(L)$ with ∞ and $(0 : 1) \in \mathbb{P}^1(L)$ with 0.)*

Proof. Choose any $M'' \in \mathrm{GL}_2(L)$ such that $M''(\infty) = w$ and $M''(0) = w'$, so that $0, \infty$ are preperiodic for $\varphi'' = M''^{-1} \circ \varphi \circ M''$. Choose any homogeneous lifting $F'' = (F_1, F_2)$ of φ'' to L^2. By Lemma 3.17,

there are lifts $(0,s),(t,0) \in L^2$ of 0 and ∞, respectively, such that $(0,s)$ and $(t,0)$ are preperiodic for F''. Let $F' = M'^{-1} \circ F'' \circ M'$, where $M' \in \mathrm{GL}_2(L)$ is the linear transformation such that $M'(x,y) = (tx, sy)$. Then $(0,1)$ and $(1,0)$ are preperiodic for F', the Möbius transformation $M = M'' \circ M'$ satisfies $M(\infty) = w$ and $M(0) = w'$, and F' is a lifting of $\varphi' = M^{-1} \circ \varphi \circ M$ as desired. □

Lemma 3.19. *Let L be an algebraically closed field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. If $g_\varphi(x,y) \geq 0$ for all $x, y \in \mathbb{P}^1(L)$, then there exist preperiodic points $w, w' \in \mathbb{P}^1(L)$ such that $g_\varphi(w,w') = 0$.*

Proof. Let $z \in \mathbb{P}^1(L)$ be a fixed point of φ, and let $w \in \mathbb{P}^1(L)$ be any periodic point of φ different from z. Let N be the period of w. By Theorem A.10, if

$$(\varphi^{(N)})^*(z) = z + \sum_{i=1}^{d^N-1} z_i'$$

as divisors on $\mathbb{P}^1(L)$, then

$$g_\varphi(w,z) = g_\varphi(\varphi^{(N)}(w), z) = g_\varphi(w,z) + \sum_{i=1}^{d^N-1} g_\varphi(w, z_i') \ . \tag{3.20}$$

Since $w \neq z$, all terms appearing in (3.20) are finite, and therefore

$$\sum_{i=1}^{d^N-1} g_\varphi(w, z_i') = 0 \ .$$

Since $g_\varphi(x,y) \geq 0$ for all $x, y \in \mathbb{P}^1(L)$ by assumption, it follows that $g_\varphi(w, z_i') = 0$ for all i. Since $\varphi^{(N)}(z_i') = z$, each z_i' is preperiodic, so we can take $w' = z_1'$. □

We are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. For the first assertion, we know that $g_\varphi(x,x) = +\infty$ for $x \in \mathbb{P}^1(L)$, so it suffices to show that if $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$, then $g_\varphi(\zeta,\zeta) \geq 0$. By Theorem 3.8, for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$ we have

$$g_\varphi(\zeta,\zeta) \geq g_\varphi(x,\zeta) \ .$$

Integrating against the probability measure μ with respect to the variable x, we obtain

$$g_\varphi(\zeta,\zeta) \geq \int_{\mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(x,\zeta)\mu(x) = 0$$

by (B3)$'$.

Now for the various equivalences. We have already remarked that $(2a) \Leftrightarrow (2b), (3a) \Leftrightarrow (3b), (4a) \Leftrightarrow (4b)$. Let us therefore refer to these assertions as $(2), (3), (4)$, respectively. We will show that $(3) \Leftrightarrow (4)$, $(4) \Leftrightarrow (2)$, and finally that $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$.

$(3) \Rightarrow (4)$:

Suppose $g_\varphi(\zeta,\zeta) = 0$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$. Then by Theorem 3.8, for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we have

$$0 = g_\varphi(\zeta,\zeta) \geq g_\varphi(x,\zeta) \ . \tag{3.21}$$

On the other hand, since $\int_{\mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(x,\zeta)\mu(x) = 0$, it follows from Lemma 3.6 that $g_\varphi(x,\zeta) = 0$ for all $x \in \mathrm{Supp}(\mu_\varphi)$. Since (by Theorem 3.8) the minimum value of $g_\varphi(x,\zeta)$ is achieved on $\mathrm{Supp}(\mu_\varphi)$, it follows from (eq:gneg) that $g_\varphi(x,\zeta) = 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. But then

$$\delta_\zeta - \mu_\varphi = \Delta_x g_\varphi(x,\zeta) = 0$$

and thus $\mu_\varphi = \delta_\zeta$.

$(4) \Rightarrow (3)$: If $\mu_\varphi = \delta_\zeta$, then since μ_φ is normalized (Corollary A.15), we have

$$0 = \iint g_\varphi(x,y)\mu_\varphi(x)\mu_\varphi(y) = g_\varphi(\zeta,\zeta) \ .$$

$(4) \Rightarrow (2)$: If $\mu_\varphi = \delta_\zeta$, then

$$g_\varphi(x,\zeta) = \int g_\varphi(x,y)\mu_\varphi(y) = 0$$

for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$ by (B3)$'$.

Fix $y \in \mathbb{P}^1_{\mathrm{Berk}}$, and let $f_y(x) = g_\varphi(x,y)$. By Theorem 3.8, the minimum value of $f_y(x)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ occurs at ζ, and therefore for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$,

$$g_\varphi(x,y) \geq g_\varphi(x,\zeta) = 0 \ .$$

$(2) \Rightarrow (4)$:

Since

$$\int g_\varphi(x,y)\mu_\varphi(y) = 0$$

for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, if $g_\varphi(x,y) \geq 0$ for all $x,y \in \mathbb{P}^1_{\mathrm{Berk}}$ and $\zeta \in \mathrm{Supp}(\mu_\varphi)$, we must have $g_\varphi(\cdot,\zeta) \equiv 0$ by Lemma 3.6. Therefore

$$\delta_\zeta - \mu_\varphi = \Delta_x g_\varphi(x,\zeta) = 0 \ ,$$

which means that $\mu_\varphi = \delta_\zeta$.

$(1) \Rightarrow (2)$:

If φ has good reduction, this follows from (3.10). The case of potentially good reduction then follows from Lemma A.3.

(4) $\Rightarrow$ (1):

Assuming that $\mu_\varphi = \delta_\zeta$, we want to show that φ has potentially good reduction.

Since (4) $\Rightarrow$ (2), Lemma 3.19 implies that we can find preperiodic points $w, w' \in \mathbb{P}^1(L)$ for φ such that $g_\varphi(w, w') = 0$. Replacing φ by a conjugate if necessary, Lemma 3.18 shows that we can assume that $w = 0, w' = \infty$, and that there is a homogeneous lifting F of φ so that $(1,0)$ and $(0,1)$ are preperiodic for F. In particular, we have $\hat{H}_F((0,1)) = \hat{H}_F((1,0)) = 0$. We will show that F has good reduction. By Corollary 3.13, it suffices to prove that $K_F = B(0,1)$.

Recall that for any lifts $z, z' \in L^2 \backslash \{0\}$ of $P, P' \in \mathbb{P}^1(L)$, respectively, we have

$$g_\varphi(P, P') = G_F(z, z') = -\log|z \wedge z'| + \hat{H}_F(z) + \hat{H}_F(z') + \log|\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}} .$$

In particular,

$$0 = g_\varphi(w, w') = G_F((0,1),(1,0)) = \log|\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}} ,$$

so that $|\operatorname{Res}(F)| = 1$ and

$$G_F(z, z') = -\log|z \wedge z'| + \hat{H}_F(z) + \hat{H}_F(z')$$

for all $z, z' \in L^2 \backslash \{0\}$.

Take $z = (x, y) \in K_F$. Then $\hat{H}_F(z) \le 0$, so condition (2) implies that

$$0 \le G_F(z, (0,1)) = -\log|x| + \hat{H}_F(z) \le -\log|x|$$

and thus $|x| \le 1$. An identical calculation using $(1,0)$ shows that $|y| \le 1$ as well. Thus $z \in B(0,1)$, and we conclude that $K_F \subseteq B(0,1)$.

Let $\varepsilon > 0$, and take $z = (x, y) \in B(0,1)$ and $z' = (x', y') \in K_F$ so that $-\varepsilon < \hat{H}_F(z') < 0$. Since $z, z' \in B(0,1)$, we have $|z \wedge z'| \le 1$, and thus

$$G_F(z, z') \ge \hat{H}_F(z) - \varepsilon . \tag{3.22}$$

Let P be the image of z in $\mathbb{P}^1(L)$, and let $P' \in \mathbb{P}^1(L)$ be arbitrary. By (3.2), we may choose a lift $z' \in L^2 \backslash \{0\}$ of P' such that $-\varepsilon < \hat{H}_F(z') < 0$. Since $g_\varphi(x, y)$ is continuous off the diagonal and $\mu_\varphi = \delta_\zeta$ implies that $g_\varphi(\zeta, y) \equiv 0$, if $P' \to \zeta$ then

$$G_F(z', z) = g_\varphi(P', P) \to g_\varphi(\zeta, P) = 0.$$

As $\mathbb{P}^1(L)$ is dense in $\mathbb{P}^1_{\mathrm{Berk}}$, we may use (3.22) to conclude that $\hat{H}_F(z) \le \varepsilon$. Letting $\varepsilon \to 0$ shows that $\hat{H}_F(z) \le 0$. Thus $z \in K_F$ and $K_F = B(0,1)$ as desired. $\square$

Remark 3.23. The fact that $g_\varphi(x,x) \geq 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, as well as the implication (3) $\Rightarrow$ (4) in the proof of Theorem 3.14, can also be deduced from the following "Energy Minimization Principle" (see [4], Theorem 7.20):

Define the "energy functional" $I_\varphi(\nu)$ on the space $\mathbb{P}$ of probability measures on $\mathbb{P}^1_{\mathrm{Berk}}$ by the formula

$$I_\varphi(\nu) \ = \ \iint_{\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(z,w)\,\nu(z)\nu(w) \ .$$

Then $I_\varphi(\nu) \geq I_\varphi(\mu_\varphi) = 0$ for all $\nu \in \mathbb{P}$, with equality if and only if $\nu = \mu_\varphi$.

As a consequence of Theorem 3.14, we present new proofs of some theorems of R. Benedetto and J. Rivera-Letelier concerning reduction of rational maps. For example, we easily deduce from Theorem 3.14 the following result of Benedetto ([7], Theorem B, see also [17], §7):

Corollary 3.24. *φ has potentially good reduction iff $\varphi^{(n)}$ has potentially good reduction for some (equivalently, every) $n \geq 1$.*

Proof. This follows from Theorem 3.14, together with the fact that

$$g_\varphi(z,w) = g_{\varphi^{(n)}}(z,w)$$

for any $n \geq 1$, which follows easily from the definition of $g_\varphi(z,w)$ using the fact that

$$|\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}} = |\operatorname{Res}(F^{(n)})|^{-\frac{1}{d^n(d^n-1)}}$$

by (A.2). $\square$

The *exceptional set* $\mathcal{E}(\varphi)$ of φ in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$, is defined to be the set of all $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$ such that

$$\bigcup_{n \geq 0} \varphi^{(-n)}(\zeta)$$

is finite. The following result is originally due to Rivera-Letelier (see [17], §7, Theorem 3):

Corollary 3.25. *If φ has genuinely bad reduction, then $\mathcal{E}(\varphi) = \emptyset$. If φ has potentially good reduction, then $\mathcal{E}(\varphi) = \{\zeta\}$ consists of a single point.*

Proof. Since φ maps $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$ surjectively onto itself, the forward orbit of any point in $\mathcal{E}(\varphi)$ must be a finite cycle of length n. Using Corollary 3.24, after replacing φ by $\varphi^{(n)}$ it suffices to prove that there is at most one point $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$ which is completely invariant under

φ, and that there exists a completely invariant point if and only if φ has potentially good reduction.

If $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$ is completely invariant, then $\varphi^*(\zeta) = d\zeta$ as divisors on $\mathbb{P}^1_{\mathrm{Berk}}$, and by Theorem A.10 we have

$$g_\varphi(\zeta, \zeta) = g_\varphi(\varphi(\zeta), \zeta) = g_\varphi(\zeta, \varphi^*(\zeta)) = d \cdot g_\varphi(\zeta, \zeta) \ ,$$

so that $g_\varphi(\zeta, \zeta) = 0$. By Theorem 3.14, this implies that φ has potentially good reduction, and that $\mu_\varphi = \delta_\zeta$. In particular, there can be at most one such point ζ. Conversely, if φ has potentially good reduction, then $\mu_\varphi = \delta_\zeta$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(L)$. Since $\varphi^*(\mu_\varphi) = d\mu_\varphi$ and $\varphi_*(\mu_\varphi) = \mu_\varphi$, it follows easily that ζ is completely invariant under φ. □

Remark 3.26. Let ζ_0 denote the *Gauss point* of $\mathbb{P}^1_{\mathrm{Berk}}$ (see [4], §1). Then it is not hard using the above considerations to show the equivalence of the following conditions:

(1) φ has good reduction.
(2) $\varphi^{(n)}$ has good reduction for some (equivalently, every) $n \geq 1$.
(3) $\varphi^{-1}(\zeta_0) = \{\zeta_0\}$.
(4) $\mu_\varphi = \delta_{\zeta_0}$.

We leave the details to the reader. The equivalence of (1) and (2) is originally due to Benedetto ([7], Theorem B), and the equivalence of (1) and (3) was first noted by Rivera-Letelier in [17], §4.1. Rivera-Letelier also shows in [17] that φ has *non-constant* reduction if and only if $\varphi(\zeta_0) = \zeta_0$.

4. A criterion for isotriviality

We recall the statement of Theorem 1.9:

Theorem. *Let K be a function field, and let $\varphi \in K(T)$ be a rational map of degree at least 2. Then φ is isotrivial if and only if φ has potentially good reduction over K_v for all $v \in M_K$.*

Proof. It follows from the definitions that isotrivial maps have everywhere potentially good reduction. We therefore assume that φ has potentially good reduction over K_v for all v, and want to show that φ is isotrivial.

Since $\mathbb{P}^1(\bar{K})$ contains infinitely many preperiodic points for φ, after conjugating φ by a Möbius transformation and replacing K by a finite extension if necessary, we may assume without loss of generality by Lemma 3.18 that there is a homogeneous lifting F of φ for which $(0, 1)$ and $(1, 0)$ are preperiodic. In particular, $P = (0, 1)$ and $Q = (1, 0)$ belong to the homogeneous filled Julia set $K_{F,v}$ for all $v \in M_K$.

Let $v \in M_K$ be a (non-archimedean) place of K, let $\mathbb{C}_v$ be the smallest complete and algebraically closed field containing the completion K_v of v, and let $\mathcal{O}_v$ be its valuation ring. Let E_v be the homogeneous filled Julia set of F in $\mathbb{C}_v^2$. By Lemma 3.16, there exists $M = M_v \in \mathrm{GL}_2(\mathbb{C}_v)$ such that $F' = M^{-1} \circ F \circ M$ has good reduction over $\mathbb{C}_v$. Write $M^{-1}(P) = P'$ and $M^{-1}(Q) = Q'$; since $E'_v = M^{-1}(E_v)$ is the homogeneous filled Julia set for F', we have $E'_v = B(0,1)$ in $\mathbb{C}_v^2$. As $aP' + bQ' \in B(0,1)$ for all $a, b \in \mathbb{C}_v$ with $|a|, |b| \leq 1$, and since M is linear and takes $B(0,1)$ to E_v, it follows that $aP + bQ \in E_v$ for all $|a|, |b| \leq 1$. Since $P = (0,1)$ and $Q = (1,0)$, we thus have $B(0,1) \subseteq E_v$.

In particular, $d^0_\infty(E_v) \geq 1$ for all $v \in M_K$. But according to (3.11) and the product formula, we have

$$\prod_{v \in M_K} d^0_\infty(E_v) \leq 1 \ .$$

Therefore $d^0_\infty(E_v) = 1$ for all $v \in M_K$.

Since $E'_v = B(0,1)$, letting d^0 denote the "wedge diameter" of a set as in §3.3, we have

$$d^0_\infty(E_v) = |\det(M)| d^0_\infty(E'_v) = |\det(M)| d^0(E'_v) = d^0(E_v)$$

for all $v \in M_K$. Thus $B(0,1) \subseteq E_v$ and $d^0(E_v) = 1$ for all $v \in M_K$. But this implies that $E_v = B(0,1)$ for all $v \in M_K$, since if $(x,y) \in E_v$, then $|x| = |(x,y) \wedge (0,1)| \leq 1$ and $|y| = |(x,y) \wedge (1,0)| \leq 1$, so that $(x,y) \in B(0,1)$.

To conclude, write $F(x,y) = (a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d, b_0 x^d + b_1 x^{d-1} y + \cdots + b_d y^d)$ and note that $E_v = B(0,1)$ implies that $F(B(0,1)) \subseteq B(0,1)$, and hence $F_1, F_2 \in \mathcal{O}_v[x,y]$ by Lemma 3.12. But then $|a_i|_v, |b_i|_v \leq 1$ for all $0 \leq i \leq d$ and all $v \in M_K$, which means that all coefficients of F lie in the constant field k_0 of K. Thus φ is isotrivial as desired. □

Appendix A. Functional equation for the dynamical Green's function and consequences

Let L be a valued field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Our goal in this appendix is to prove a functional equation for the dynamical Green's function $g_\varphi(z,w)$ (Theorem A.10), and to deduce some consequences of this formula. Some of these results are also proved in [5] in the case where L is the completion of a global field. The proofs given here, by contrast, are purely local.

Let k be a field. By [15], Theorem IX.3.13, if $F = (F_1, F_2)$ and $G = (G_1, G_2)$ where $F_1, F_2 \in k[X,Y]$ are homogeneous of degree d and

$G_1, G_2 \in k[X,Y]$ are homogeneous of degree e, then

$$\mathrm{Res}(F \circ G) = \mathrm{Res}(F)^e \, \mathrm{Res}(G)^{d^2}. \tag{A.1}$$

In particular, it follows by induction on n that

$$\mathrm{Res}(F^{(n)}) = \mathrm{Res}(F)^{\frac{d^{n-1}(d^n-1)}{d-1}} \ . \tag{A.2}$$

Lemma A.3. *Let L be a valued field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Let F be a homogeneous lifting of φ, let $M \in \mathrm{GL}_2(L)$, and let $F' = M^{-1} \circ F \circ M$. Then for all $z, w \in L^2 \backslash \{0\}$,*

$$\hat{H}_F(M(z)) = \hat{H}_{F'}(z) \tag{A.4}$$

and

$$G_F(M(z), M(w)) = G_{F'}(z, w) \ . \tag{A.5}$$

Proof. First of all note that given M, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|z\| \leq \|M(z)\| \leq C_2 \|z\| \ . \tag{A.6}$$

Indeed, if we take C_2 to be the maximum of the absolute values of the entries of M then clearly $\|M(z)\| \leq C_2 \|z\|$ for all z. By the same reasoning, if we let C_1^{-1} be the maximum of the absolute values of the entries of M^{-1}, then

$$\|M^{-1}(M(z))\| \leq C_1^{-1} \|M(z)\| \ ,$$

which gives the other inequality.

By the definition of $\hat{H}_F$, we have

$$\begin{aligned} \hat{H}_F(M(z)) &= \lim_{n\to\infty} \frac{1}{d^n} \log \|F^{(n)}(M(z))\| \\ &= \lim_{n\to\infty} \frac{1}{d^n} \log \|M F'^{(n)}(z)\| = \hat{H}_{F'}(z) \ , \end{aligned}$$

where the last equality follows from (A.6). This proves (A.4).

Since $|M(z) \wedge M(w)| = |\det(M)| \cdot |z \wedge w|$, we have

$$-\log|M(z) \wedge M(w)| = -\log|\det(M)| - \log|z \wedge w| \ . \tag{A.7}$$

On the other hand, by (A.1) and the fact that $|\mathrm{Res}(M)| = |\det(M)|$, it follows that

$$-\frac{1}{d(d-1)} \log|\mathrm{Res}(F')| = -\frac{1}{d(d-1)} \log|\mathrm{Res}(F)| - \log|\det(M)| \ . \tag{A.8}$$

Putting together (A.4),(A.7), and (A.8) gives (A.5). □

Corollary A.9. *Let L be a valued field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Let $M \in \mathrm{PGL}_2(L)$. Then for all $z, w \in \mathbb{P}^1(L)$, we have*

$$g_\varphi(M(z), M(w)) = g_{M^{-1}\circ\varphi\circ M}(z, w) \ .$$

The following result gives a useful functional equation for the dynamical Green's function. A proof using the fact that g_φ is *normalized* (i.e., satisfies condition (B3) above) is given in [5], Corollary 3.39. However, the proof in [5] that g_φ is normalized uses global methods. Here, we give a simple, purely local proof of the functional equation, and instead deduce from the functional equation that g_φ is normalized (Corollary A.15).

Theorem A.10. *Let L be an algebraically closed valued field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Then for all $x, y \in \mathbb{P}^1(L)$, we have*

$$g_\varphi(\varphi(x), y) = g_\varphi(x, \varphi^*(y)) \ ,$$

where $y_1, \ldots, y_d$ are the preimages of y under φ, counting multiplicity, and

$$g_\varphi(x, \varphi^*(y)) := \sum_{i=1}^{d} g_\varphi(x, y_i) \ .$$

Proof. Using Corollary A.9, we may assume without loss of generality that $y = \infty$. Let F be a homogeneous lifting of φ, and let $R = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}}$. For $z \in L^2\backslash\{0\}$, let $[z]$ denote the class of z in $\mathbb{P}^1(L)$. Fix $w = (1, 0)$, let $a_1, \ldots a_d$ be the preimages of $[w]$ under φ (counting multiplicities), and let $w_1, \ldots, w_d$ be solutions to $F(w_i) = w$ such that $[w_i] = a_i$.

It suffices to show that for all $z \in L^2\backslash\{0\}$, we have

$$G_F(F(z), w) = \sum_{i=1}^{d} G_F(z, w_i) \ . \tag{A.11}$$

Since $\hat{H}_F(F(z)) = d\hat{H}_F(z)$ and $\hat{H}_F(w_i) = \frac{1}{d}\hat{H}_F(w)$ for all i, (A.11) is equivalent to

$$-\log|F(z) \wedge w| = (d-1)\log R - \sum_{i=1}^{d} \log|z \wedge w_i| \ ,$$

which itself is equivalent to

$$|F(z) \wedge w| = |\operatorname{Res}(F)|^{1/d} \prod_{i=1}^{d} |z \wedge w_i| \ . \tag{A.12}$$

We verify (A.12) by an explicit calculation (compare with [11], Lemma 6.5).

Write $F = (F_1, F_2) = (\prod_{i=1}^d z \wedge a_i, \prod_{j=1}^d z \wedge b_j)$. Since $F(b_j) = (\prod_{i=1}^d a_i \wedge b_j, 0)$ and $w = (1, 0)$, we may take

$$w_j = \frac{b_j}{(\prod_i a_i \wedge b_j)^{1/d}}$$

for $j = 1, \ldots, d$, where for each j we fix some choice of a dth root of $\prod_i a_i \wedge b_j$. Note that $|w_j|$ is independent of which dth root we pick. Thus

$$\prod_{j=1}^d |z \wedge w_j| = \frac{\prod_j |z \wedge b_j|}{\prod_{i,j} |a_i \wedge b_j|^{1/d}} = \frac{|F(z) \wedge w|}{|\operatorname{Res}(F)|^{1/d}} ,$$

which gives (A.12). □

Suppose now that L is a complete and algebraically closed *non-archimedean* field, and let $\mathbb{P}^1_{\text{Berk}}$ denote the Berkovich projective line over L. As a function on $\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}$, $g_\varphi(z, w)$ is lower semicontinuous, continuous off the diagonal, and symmetric in z and w. Also, for each fixed $z \in \mathbb{P}^1_{\text{Berk}}$, the function $w \mapsto g_\varphi(z, w)$ is continuous on $\mathbb{P}^1_{\text{Berk}} \backslash \{z\}$ (and is continuous and real-valued on all of $\mathbb{P}^1_{\text{Berk}}$ if $z \notin \mathbb{P}^1(L)$). Moreover, $\varphi : \mathbb{P}^1_{\text{Berk}} \to \mathbb{P}^1_{\text{Berk}}$ is continuous, and both $\mathbb{P}^1(L)$ and $\mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$ are dense in $\mathbb{P}^1_{\text{Berk}}$. (See [4], §7, for proofs of all of these assertions.)

In addition, it is shown in [4], §7, that for each $y \in \mathbb{P}^1_{\text{Berk}}$, there is a natural way to define the pullback $\varphi^*(y) = \sum m_i x_i$, with $\varphi(x_i) = y$ and $1 \leq m_i \leq d$ the *multiplicity* of x_i, so that $\sum m_i = d$ and for any continuous function $f : \mathbb{P}^1_{\text{Berk}} \to \mathbb{R}$, the function $\varphi_*(f)$ defined by

$$\varphi_*(f)(y) = \sum_{\varphi(x)=y} m_x f(x)$$

is again continuous. For $y \in \mathbb{P}^1(L)$, m_i is just the usual algebraic multiplicity of x_i as a solution to $\varphi(\cdot) = y$.

Using these facts, we can deduce that Theorem A.10 remains valid for all $x, y \in \mathbb{P}^1_{\text{Berk}}$:

Corollary A.13. *Let L be a complete and algebraically closed non-archimedean field, and let $\varphi \in L(T)$ be a rational map of degree $d \geq 2$. Then for all $x, y \in \mathbb{P}^1_{\text{Berk}}$, we have*

$$g_\varphi(\varphi(x), y) = g_\varphi(x, \varphi^*(y)) \ . \tag{A.14}$$

Proof. Fix $y \in \mathbb{P}^1(L)$. Then since φ is continuous, both $g_\varphi(\varphi(x), y)$ and $g_\varphi(x, \varphi^*(y))$ are continuous real-valued functions of x for all $x \in$

$\mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$. Since these functions agree on the dense subset $\mathbb{P}^1(L)$ of $\mathbb{P}^1_{\text{Berk}}$ by Theorem A.10, (A.14) holds for all $x \in \mathbb{P}^1_{\text{Berk}}, y \in \mathbb{P}^1(L)$.

Now fix $x \in \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$. Then $g_\varphi(\varphi(x), y)$ and $\varphi_* g_\varphi(x, y)$ are continuous and real-valued for all $y \in \mathbb{P}^1_{\text{Berk}}$, and they agree for $y \in \mathbb{P}^1(L)$. Therefore (A.14) holds for all $x \in \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L), y \in \mathbb{P}^1_{\text{Berk}}$.

Finally, fix $y \in \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$. Then $g_\varphi(\varphi(x), y)$ and $g_\varphi(x, \varphi^*(y))$ are continuous and real-valued for all $x \in \mathbb{P}^1_{\text{Berk}}$, and they agree on the dense subset $\mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(L)$, so they agree for all $x \in \mathbb{P}^1_{\text{Berk}}$. □

Using Corollary A.13, we can deduce that g_φ is a *normalized* Arakelov-Green's function on $\mathbb{P}^1_{\text{Berk}}$. The proof remains valid for any complete and algebraically closed field L if we define $\mathbb{P}^1_{\text{Berk}}$ to be $\mathbb{P}^1(L)$ when L is archimedean.

Corollary A.15. *For all $x \in \mathbb{P}^1_{\text{Berk}}$, we have*

$$\iint_{\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}} g_\varphi(x, y) \mu_\varphi(x) \mu_\varphi(y) = 0 \ .$$

Proof. By (3.4), there is a constant C such that $\int g_\varphi(x, y)\, \mu_\varphi(y) = C$ for all $x \in \mathbb{P}^1_{\text{Berk}}$, and it suffices to show that $C = 0$.

Fix $x \in \mathbb{P}^1_{\text{Berk}}$. Since $\varphi_* \mu_\varphi = \mu_\varphi$ (see [4], Theorem 7.14), it follows from Corollary A.13 that

$$\begin{aligned} C &= \int g_\varphi(x, y)\, \mu_\varphi(y) = \int g_\varphi(x, y)\, \varphi_* \mu_\varphi(y) \\ &= \int g_\varphi(x, \varphi(y))\, \mu_\varphi(y) \\ &= \int g_\varphi(\varphi^*(x), y)\, \mu_\varphi(y) \\ &= dC \ , \end{aligned}$$

and thus $C = 0$ as desired. □

Finally, we deduce a formula for the homogeneous transfinite diameter of homogeneous filled Julia sets:

Corollary A.16. *Let $F : L^2 \to L^2$ be a non-degenerate homogeneous polynomial map, where L is a complete and algebraically closed valued field. Let K_F be the homogeneous filled Julia set of F. Then*

$$d^0_\infty(K_F) = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}} \ .$$

Proof. (Compare with [5], Theorem 3.16, and [11], Theorem 1.5.) Let $R = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}}$, and define

$$D_n = \inf_{P_1,\ldots,P_n \in \mathbb{P}^1(L)} \frac{1}{n(n-1)} \sum_{i \neq j} g_\varphi(P_i, P_j) \ .$$

By Theorem 3.49 of [5],

$$\lim_{n\to\infty} D_n = \iint g_\varphi(x,y)\mu_\varphi(x)\mu_\varphi(y) \ ,$$

which is zero by Corollary A.15.

By (3.2), for every $P \in \mathbb{P}^1(L)$ and each $\varepsilon > 0$, there exists a lifting z of P to $L^2\backslash\{0\}$ such that $-\varepsilon \leq \hat{H}_F(z) \leq 0$ (and in particular, $z \in K_F$).

Using this observation, it follows from the definitions that for every $n \geq 2$ and every $\varepsilon > 0$,

$$-\log d_n^0(K_F) + \log R - 2\varepsilon \leq D_n \leq -\log d_n^0(K_F) + \log R \ .$$

Letting $\varepsilon \to 0$ and $n \to \infty$ gives the desired result. □

Appendix B. The Mordell-Weil theorem for elliptic curves over function fields

In this section, we give a new proof of the Mordell-Weil theorem for elliptic curves over function fields. In addition to its simplicity and quantitative nature, our proof illustrates many of the salient features in the proof of Theorem 1.6 without the complications introduced by the theory of Berkovich spaces.

Let K be the function field of a nonsingular, integral projective algebraic curve C over an algebraically closed field k_0, so that k_0 is the field of constants of K.

Let E/K be an elliptic curve. We will say that E is *isotrivial over* K if there is an elliptic curve E_0/k_0 and a K-isomorphism from E to E_0, and *isotrivial* if there exists a finite extension K'/K and an elliptic curve E' which is isotrivial over K' such that E is isomorphic to E' over K'.

The following result should be compared with Theorem 1.9.

Lemma B.1. *E is isotrivial if and only if E has potentially good reduction at all $v \in M_K$.*

Proof. Let $j_E \in K$ denote the j-invariant of E. By [19], Proposition VIII.5.5, E has potentially good reduction at v if and only if $|j_E|_v \leq 1$. On the other hand, by [19], Proposition III.1.4, we see that E is isotrivial if and only if $j_E \in k_0$. The result now follows from the definition of k_0. □

Let $\hat{h}_E = \hat{h}_{E,K} : E(K) \to \mathbb{R}$ be the Néron-Tate canonical height on E. (If we fix a Weierstrass equation $y^2 = f(x)$ for E over K and let φ be the degree four rational map expressing $x(2P)$ in terms of $x(P)$, then $\hat{h}$ can be defined by the formula $\hat{h}_E(P) = \frac{1}{2}\hat{h}_\varphi(x(P))$.)

According to Néron's theory (see [20], Chapter VI, and [18], §6.5), the global canonical height $\hat{h}_E$ can be decomposed into a sum of canonical local heights: for $P \in E(K), P \neq 0$, we have

$$\hat{h}_E(P) = \sum_{v \in M_K} \lambda_v(P) \ , \tag{B.2}$$

where the functions $\lambda_v : E(\mathbb{C}_v)\backslash\{0\} \to \mathbb{R}$ satisfy

(1) If E has good reduction at v, then $\lambda_v(P) \geq 0$ for all $P \in E(\mathbb{C}_v)\backslash\{0\}$.
(2) If E has split multiplicative reduction at v, then $\lambda_v(P) = i(P) + j(P)$ for all $P \in E(\mathbb{C}_v)\backslash\{0\}$, where $i(P) \geq 0$ and $j(P) = \frac{1}{2}v(q)\mathbb{B}_2(r(P))$. Here $v(q) = -\log|j_E|_v > 0$ (for an appropriate choice of logarithm), $r : E(\mathbb{C}_v) \to \mathbb{R}/\mathbb{Z}$ is the "retraction homomorphism" described in [3], §3.1, and $\mathbb{B}_2(t) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the periodic second Bernoulli polynomial defined for $0 \leq t \leq 1$ by $\mathbb{B}_2(t) = t^2 - t + 1/6$.

By the parallelogram law,

$$\hat{h}_E(P-Q) = 2\hat{h}_E(P)+2\hat{h}_E(Q)-\hat{h}_E(P+Q) \leq 2\hat{h}_E(P)+2\hat{h}_E(Q) \ . \tag{B.3}$$

Also, since $\frac{1}{2}\mathbb{B}_2(0) = \frac{1}{12} > 0$, it follows by continuity that the circle $\mathbb{R}/\mathbb{Z}$ can be decomposed as a union of finitely many segments $U_1, \ldots, U_s$ such that $\mathbb{B}_2(x-y) > 0$ whenever $x, y \in U_i$ for some i. For example, we can take $U_i = [\frac{i-1}{6}, \frac{i}{6}]$ for $i = 1, \ldots, 6$, in which case

$$\frac{1}{2}\mathbb{B}_2(x-y) \geq \frac{1}{72} \tag{B.4}$$

for $x, y \in U_i$, $1 \leq i \leq 6$.

Finally, a simple Fourier averaging argument ([20], Exercise 6.11) shows that if $x_1, \ldots, x_N \in \mathbb{R}/\mathbb{Z}$, then

$$\sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} \frac{1}{2}\mathbb{B}_2(x_i - x_j) \geq -\frac{N}{12} \ . \tag{B.5}$$

We now prove:

Theorem B.6. *If E/K is not isotrivial, then there is a constant $\varepsilon > 0$ depending on E and K such that the set $\{P \in E(K) : \hat{h}_E(P) \leq \varepsilon\}$ is finite.*

Proof. Replacing K by a finite extension, we may assume without loss of generality that E has semistable reduction over K, and thus that E has either good or split multiplicative reduction at each $v \in M_K$. By Lemma B.1, the set S of places of bad reduction for E is non-empty; let s be its cardinality. For each $v \in S$, let $\delta_v = -\frac{1}{12}\log|j_E|_v > 0$, let $\delta = \min_{v \in S}\{\delta_v\}$, and let $v_0 \in S$ be a place for which $\delta_{v_0} = \delta$. Finally, let $S' = S \backslash \{v_0\}$.

Fix $N \geq 2$, let $M = 6(N-1)+1$, and suppose that $P_1, \ldots, P_M \in E(K)$ are distinct points each having $\hat{h}_E(P_i) < \delta/48$.

Iterating (B.3) and letting

$$g_v(P_1, \ldots, P_N) = \frac{1}{N(N-1)} \sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} \lambda_v(P_i - P_j) \ ,$$

we obtain

$$\sum_{v \in M_K} g_v(P_1, \ldots, P_N) \leq \frac{4}{N} \sum_i \hat{h}_E(P_i) < \frac{\delta}{12} \ . \tag{B.7}$$

On the other hand, by (B.4) and the pigeonhole principle, after relabelling the P_i's we may assume that $\lambda_{v_0}(P_i - P_j) \geq \delta/6$ for all $1 \leq i, j \leq N$. Thus

$$g_{v_0}(P_1, \ldots, P_N) \geq \delta/6 \ .$$

In addition, for $v \in S'$, (B.5) gives

$$g_v(P_1, \ldots, P_N) \geq -\frac{\delta_v}{N-1} \ .$$

If $N \geq 12(s-1)+1$, it follows that

$$\begin{aligned} \sum_{v \in M_K} g_v(P_1, \ldots, P_N) &\geq g_{v_0}(P_1, \ldots, P_N) + \sum_{v \in S'} g_v(P_1, \ldots, P_N) \\ &\geq \frac{\delta}{6} - \frac{\delta}{12} \\ &= \frac{\delta}{12} \ , \end{aligned}$$

contradicting (B.7).

Thus $N \leq \max\{2, 12(s-1)\}$, and therefore $M \leq \max\{7, 72(s-1)-5\}$. In other words, there are at most $\max\{7, 72(s-1)-5\}$ distinct points in $E(K)$ with height less than $\delta/48$. $\square$

We recall from [14], Proposition 6.2.3, (see also [20], Theorem III.2.1, when $\mathrm{char}(K) = 0$) the following "weak Mordell-Weil theorem" over

K; its proof is more or less the same as in the case where K is a number field.

Theorem B.8. *If E/K is not isotrivial and $E[m](\bar{K}) \subseteq E(K)$, then for every integer $m \geq 1$ not divisible by the characteristic of K, the group $E(K)/mE(K)$ is finite.*

According to Lemma 1.4, for M sufficiently large there are infinitely many points $x \in \mathbb{P}^1(K)$ such that $\hat{h}_\varphi(x) \leq M$. However, most of these points do not lift to K-rational points of E, as the following result shows:

Theorem B.9. *If E/K is not isotrivial, then for every $M > 0$, the set*

$$\{P \in E(K) \; : \; \hat{h}_E(P) \leq M\}$$

is finite.

Proof. (Compare with the proof of [13], Lemma 2.2.)

Let $m \geq 2$ be any integer not divisible by $\mathrm{char}(K)$. Without loss of generality, we may replace K by a finite extension and assume that $E[m](\bar{K}) \subseteq E(K)$. Now suppose the result in question is false, and let

$$C = \inf\{M \; : \; \exists \text{ infinitely many } P \in E(K) \text{ with } \hat{h}_E(P) \leq M\} \; .$$

By Theorem B.6, $C > 0$. By the definition of C, there exists an infinite sequence $P_n \in E(K)$ such that $\hat{h}_E(P_n) < 3C/2$ for all n. Since $E(K)/mE(K)$ is finite by Theorem B.8, there is a coset of $mE(K)$ in $E(K)$ containing infinitely many of the points P_n. But if $i \neq j$ and P_i and P_j are in the same coset, then $P_i - P_j = mQ$ for some $Q \in E(K)$, and it follows from the parallelogram law that

$$\hat{h}_E(Q) \leq \frac{\hat{h}_E(P_i) + \hat{h}_E(P_j)}{m^2} < \frac{3C}{4} < C \; .$$

It follows that there are infinitely many $Q \in E(K)$ such that $\hat{h}_E(Q) < \frac{3C}{4}$, contradicting our choice of C. □

Using Theorems B.8 and B.9, the usual descent argument (see [19], Proposition VIII.3.1) now establishes the following version of the Mordell-Weil theorem:

Corollary B.10 (Mordell-Weil Theorem)**.** *If E/K is not isotrivial, then $E(K)$ is finitely generated.*

Remark B.11.

(i) Theorem B.9 and Corollary B.10 are proved in [14], Chapter 5, in any characteristic under the weaker hypothesis that E is not isotrivial *over K*.

(ii) Unlike the proofs in [14] or [20], our proof of Theorem B.9 remains valid over an arbitrary function field K (in the general sense discussed in the introduction).

Finally, we sketch how one can deduce Theorem B.6 directly from Theorem 1.6. For simplicity, we assume that K has characteristic different from 2,3.

Fix a minimal Weierstrass equation $y^2 = x^3 + Ax + B$ for E over K_v with discriminant Δ, and recall that the Néron-Tate canonical height $\hat{h}_E = \hat{h}_{E,K} : E(K) \to \mathbb{R}$ on E can be defined for $P = (x(P), y(P)) \in E(K)$ as

$$\hat{h}_E(P) = \frac{1}{2}\hat{h}_\varphi(x(P))$$

where φ is a degree 4 rational function defined over K.

If $x(P) = (z_0, z_1) \in \mathbb{P}^1(K)$, we have $\varphi(x(P)) = x([2]P) = (F_1(z_0, z_1) : F_2(z_0, z_1))$ with $F_1(X,Z), F_2(X,Z) \in \mathcal{O}_v[X,Z]$ homogeneous of degree 4. More concretely, according to [19], § III.2.3d, we have

$$\begin{aligned} F_1(X,Z) &= X^4 - 2AX^2Z^2 - 8BX + A^2 \ , \\ F_2(X,Z) &= 4X^3Z + 4AXZ^3 + 4BZ^4 \ . \end{aligned}$$

An explicit computation shows that $\mathrm{Res}(F_1(X,Z), F_2(X,Z)) = 2^8\Delta^2$. In particular, since K_v has residue characteristic different from 2, we see that

$$|\,\mathrm{Res}(F_1, F_2)|_v = |\Delta|_v^2 \ . \tag{B.12}$$

It is not hard to show that for all $P, Q \in E(\mathbb{C}_v)$, λ_v and $g_{\varphi,v}$ are related by the formula

$$\lambda_v(P+Q) + \lambda_v(P-Q) = g_{\varphi,v}(x(P), x(Q)) \ . \tag{B.13}$$

[**Sketch of proof:** One checks that for fixed w, $g_{\varphi,v}(z,w)$ is a Call-Silverman canonical local height function on $\mathbb{P}^1$ relative to the divisor (w); this implies by functoriality of canonical local heights that both sides of (B.13) are equal up to a constant. Finally, one shows that the constant is zero using (B.12).]

In particular,

$$\lambda_v(P) = \frac{1}{2} g_{\varphi,v}(x(P), \infty) \ . \tag{B.14}$$

Lemma B.15. *E has potentially good reduction at $v \in M_K$ if and only if φ does.*

Proof. Replacing K by a finite extension, we may assume without loss of generality that E has semistable reduction over K. The explicit formulas for λ_v then show that $\lambda_v(P) \geq 0$ for all $P \in E(\mathbb{C}_v)$ if and only

if E has good reduction at v. On the other hand, Theorem 3.14 shows that $g_v(z,w) \geq 0$ for all $z,w \in \mathbb{P}^1(\mathbb{C}_v)$ if and only if φ has potential good reduction at v. The result now follows easily from (B.13) and (B.14), together with the fact that the map $x : E(\mathbb{C}_v) \to \mathbb{P}^1(\mathbb{C}_v)$ is surjective. $\square$

Theorem B.6 is now an immediate consequence of Theorem 1.6 and Lemmas B.1 and B.15.

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